On the Linkage Effect in the Paradigm of Affiliated Signals and Interdependent Values*

Melody Lo, Artyom Shneyerov, Pai Xu

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Abstract

The digital platforms widely connect population and offer great opportunities for transactions. Ample opportunities of resale provide common-value elements to all transaction parties. Ease of information acquisition regarding this common-value aspect makes buyers and sellers more correlated in the information they individually possess. We therefore study a second-price auction model with both interdependent values and affiliated signals to capture the feature of digital markets.

Interdependence in our model suggests participation of another bidder raises the price due to the information her presence reveals. Such an information channel across bidders gives rise to a *linkage effect* that incentivises the seller to encourage entry. Since the linkage effect goes in the opposite direction than the information rent, it therefore offers a device to potentially correct the issue of under-provision by monopolists.

We examine the conditions for existence and uniqueness of a separating equilibrium where higher reserve price signals more favourable seller information. In our environment, a seller faces the trade-off as follows. On one hand, sellers with better signals believe that buyers tend to have more favourable information as well, due to affiliated signals. This implies a stronger linkage effect, which in turn incentivises the seller to lower the reserve price to encourage entry; On the other hand, however, a separating equilibrium requires seller with higher signals to set higher reserve prices. We show that a separating equilibrium exists when the linkage effect increases with sellers' signals at a moderate rate. This is the case when neither value interdependence and signal affiliation is too strong.

Keywords: Auctions, affiliation, signalling, Spence-Mirrlees condition, Myerson's virtual value

JEL Codes: D82, D83

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1 Introduction

A *reserve price* is one of the main revenue tools available to the auction seller. The choice of the optimal reserve price is therefore a fundamental question in auction design beginning with Myerson (1981) and Riley and Samuelson (1981). The earlier literature assumed independent private values, but in many applications the values are better modelled as *interdependent*, i.e. both the seller and the buyers may have information relevant to the other party. This brings the question of information transmission in the auction through the reserve price in a separating equilibrium front and center. Jullien and Mariotti (2006), Cai, Riley, and Ye (2007) and Lamy (2010) have previously investigated reserve price signalling in second-price auctions where the seller's value is independent of buyers' values.[1] Cai, Riley, and Ye (2007) provided a general condition for existence of a separating equilibrium in this model. Their condition essentially amounts to the monotonicity of the Myerson (1981) virtual value.[2]

An important limitation of the aforementioned papers is the assumption that buyers' and the seller's signals are *independent*, so the seller's information does not affect her assessment about the buyers' beliefs. It rules out empirically relevant environments with a common value component about which both buyers and the seller receive informative signals. Consider, for example, bidding in eBay auctions, where the sellers often post a reserve price. For items such as collectibles or cars, the seller knows the actual condition of the item, and this makes his information relevant to the buyers. Buyers, by carefully inspecting the listing, may also obtain information relevant to themselves and other buyers.[3] In addition, when both buyers and the seller collect information about some common element, their information are naturally correlated. For example, collectible items on eBay are often bought by dealers with the goal of resale, and the seller can easily run another auction on potentially a different platform. If the seller believes that the item is likely to fetch a high price in another auction because of favorable market assessment, the seller tends to believe that the dealers among the buyers also hold favorable information about the item's resale value. This leads to affiliation between the seller and the buyers' information.

[1] In a general mechanism design setting, Cremer and McLean (1988) have shown that an incentive compatible and interim individually rational mechanism exists that fully extracts the buyer's surplus.

[2] Lamy (2010) has obtained a corrected version of the condition.

[3] See e.g. Bajari and Hortacsu (2003)..

In this paper, we obtain a sufficient condition for a separating equilibrium in this general interdependent value model assuming that the buyer and seller signals are *affiliated*, as in Milgrom and Weber (1982). As in any signalling game, the key condition for a separating equilibrium is the single crossing condition. Our result shows that this condition, together with the Myerson virtual profit monotonicity, essentially ensures existence, and in fact uniqueness, of the separating equilibrium.[4]

In the context of our model, the Spence-Mirrlees condition turns out to be algebraically involved. Therefore, it is desirable to have a simpler condition. We show that in this model, the Spence-Mirrlees condition holds if virtual profit is decreasing in seller type, making higher seller types less willing to sell. This condition (and therefore the Spence-Mirrlees) is automatically satisfied in Cai, Riley and Ye (2007)'s model where the seller's and buyers' signals are independent. We further show that it is satisfied if the combination of interdependent value and affiliated signals is not too strong, for example when either the buyers' values are independent conditional on the seller's signal (NO interdependent value), or if the buyers' signals are independent of the seller's signal (NO affiliated signals).

The focus of information transmission by sellers connects this paper with previous works in technical aspects. Riley (1979) and Mailath (1987) both study separating equilibria where the seller signals her type. However, some assumptions needed for the existence of such an equilibrium fail to hold in our setting. For example, seller's profit may not increase strictly with belief in our model, as trade may not always happen. Meanwhile, we do not assume any seller type, particularly good ones, will surely trade. This violates the environments in Riley (1979) that the seller prefers pooling with the lowest type than separating herself through the highest signal. We are able to relax these assumptions by studying a vector field consistent with the differential equation defined by the problem, instead of solving the differential equation directly.

In an auction environment, Jullien and Mariotti (2006) considers a specialized pay-off function with independent signals and provides characterization of separating equilibrium in such a

[4] Cai, Riley and Ye (2007) consider Myerson's virtual value, while we need to consider the virtual profit. This is because we allow the seller's value to depend on buyer signals. There are several other assumptions that are needed, either technical like differentiability of valuations, or commonly used assumptions like the monotonicity of valuations in signals.

case. We both face a technical challenge, the existence of a solution to the differential equation on the entire seller type space even though the differential equation is not well-defined on the boundary of the domain. We offers an alternative approach in this paper. In addition, we do not require the monotonicity of the marginal costs for signalling with respect to seller type in our equilibrium.

2 Model

There is a seller and n buyers. The seller has a unit of a good to sell, and each buyer has a unit demand. The seller conducts a second-price auction with a reserve price. Before the game begins, the buyers and the seller receive *signals* X_i and X_S that are informative about their ex-post valuations of the good. The signals will be sometimes referred to as the buyer's and seller's *types*. The buyers and the seller do not observe the signals received by the others, only their own signals.

Assumption 1 (Signal Affiliation)**.** $(X_1, ..., X_n, X_S)$ *are affiliated and their joint probability density function is symmetric in the first n arguments, continuously differentiable and positive on* $[0, 1]^{n+1}$.

Buyer i's valuation is given by $V_i = v_B (x_i, x_{-i}, x_S)$ and the seller's valuation is given by $V_S = v_S (x_1, ..., x_n, x_S)$. Let $X_{(k)}$ denote the k^{th} highest signal among all buyers, and $\tilde{X}_{(1)}^{-i}$ denote the highest signal among all but buyer i 's signals. We define relevant value functions as follows.

$$\begin{aligned} v(x, x_S) &= E\left[V_i | X_i = x, X_{(1)}^{-i} = x, X_S = x_S\right], \\ w(x, x_S) &= E\left[V_i | X_i = x, X_{(1)}^{-i} < x, X_S = x_S\right], \\ u_S(x, x_S) &= E\left[V_S | X_{(1)} = x, X_S = x_S\right]. \end{aligned}$$

We allow a general framework of interdependent values in our analysis where v and w can have different values. Note that v and w are equal in the special case of independent values as in Cai et al (2007).

Assumption 2 (Informative Signals)**.** *The functions $w(x, x_S)$, $v(x, x_S)$ and $u_S(x, x_S)$ are twice continuously differentiable, with*

$$\begin{aligned} \frac{\partial w}{\partial x} > 0, &\quad \frac{\partial w}{\partial x_S} > 0, \\ \frac{\partial v}{\partial x} > 0, &\quad \frac{\partial v}{\partial x_S} > 0, \\ \frac{\partial u_S}{\partial x} \geq 0, &\quad \frac{\partial u_S}{\partial x_S} > 0. \end{aligned}$$

The inequalities in the left column above assume buyers' signals matter to their own valuations, and possibly to the seller's as well. The inequalities on the right column require that the seller's information positively affects both the buyers' and her own valuations.

Let $\hat{x}_S$ denote buyer's belief about the seller type, and p the reserve price set by the seller. We call the *marginal* buyer as the lowest buyer type x that can obtain a non-negative payoff conditional on winning. This marginal buyer is defined as the unique solution to

$$w(x, \hat{x}_S) = p. \tag{1}$$

The marginal buyer can only win the auction with positive probability if she is the solo participant. In that case, she will have to recover the payment from her valuation, conditional on all her rivals' signals being below.[5] Equation (1) defines a one-to-one correspondence between the price p and the marginal buyer type x. When there are two or more participating bidders, the price is equal to the auction price $v(\tilde{x}, \hat{x}_S)$, where $\tilde{x}$ is the second-highest signal.

Let $F_{(k)}(.|x_S)$ be the CDF of the k^{th} highest buyer signal conditional on seller signal x_S, and $f_{(k)}(.|x_S)$ its probability density function. A key element in our investigation will be the expected profit of the seller x_S, given that the buyers' belief is $\hat{x}_S$ and the marginal buyer is x:

$$\begin{aligned} \pi(x, x_S, \hat{x}_S) \equiv w(x, \hat{x}_S)\left(F_{(2)}(x|x_S) - F_{(1)}(x|x_S)\right) + \int_x^1 v(\tilde{x}_B, \hat{x}_S) f_{(2)}(\tilde{x}_B|x_S)\, d\tilde{x}_B \\ + \int_0^x u_S(\tilde{x}_B, x_S) f_{(1)}(\tilde{x}_B|x_S)\, d\tilde{x}_B \end{aligned} \tag{2}$$

The first term in the above equation corresponds to a sale at the reserve price when only one bidder participates; the second term corresponds to a sale at the auction price when at least

[5] Similar arguments can be found in Milgrom and Weber (1982). See the discussion on pp.1111-1112 for more details.

two bidders participate; and the last term gives the seller's utility from keeping the good when no bidder participates. As in Cai et al (2007), the separating equilibrium in our study will be obtained in terms of x rather than p, since there is one-to-one correspondence between the reserve price and the marginal buyer x.

The *marginal profit* of the seller can be written as

$$\frac{\partial \pi\left(x, x_S, \hat{x}_S\right)}{\partial x} = -J(x, x_S, \hat{x}_S) f_{(1)}(x|x_S) \tag{3}$$

where we denoted à-la Myerson virtual profit as

$$J\left(x, x_S, \hat{x}_S\right) \equiv w\left(x, \hat{x}_S\right) - \underbrace{\frac{\partial w\left(x, \hat{x}_S\right)}{\partial x} \frac{F_{(2)}\left(x|x_S\right) - F_{(1)}\left(x|x_S\right)}{f_{(1)}\left(x|x_S\right)}}_{\text{information rent}} + \underbrace{\left(v\left(x, \hat{x}_S\right) - w\left(x, \hat{x}_S\right)\right) \frac{f_{(2)}\left(x|x_S\right)}{f_{(1)}\left(x|x_S\right)}}_{\text{linkage effect}} - u_S(x, x_S) \tag{4}$$

$J\left(x, x_S, \hat{x}_S\right)$ can be viewed as the marginal profit as if the seller lowers reserve price to sell additional unit. Therefore, all effects in the expression of $J\left(x, x_S, \hat{x}_S\right)$ are normalized by $f_{(1)}\left(x|x_S\right)$, the probability of switching from no sale to sale at reserve when marginal buyer type is reduced slightly from x. In the linkage effect term, $\frac{f_{(2)}(x|x_S)}{f_{(1)}(x|x_S)}$ is thus the probability ratio of the aforementioned two events: the probability when reduction in marginal type leads to the first participant v.s. the probability when the same reduction leads to the second participant.

As the marginal type is reduced slightly from x, two possible changes might happen. First, probability mass of $f_{(1)}\left(x|x_S\right)$ is transferred from no sale at all to sale at reserve because the number of participating bidders changes from 0 to 1. From this new marginal type, the seller obtains the difference between the reserve price and her own use value: $w\left(x, \hat{x}_S\right) - u_S\left(x, x_S\right)$. However, for this new marginal type to be willing to participate, reserve price is reduced by $\frac{\partial w(x, \hat{x}_S)}{\partial x}$ for all participating types, i.e. all types above x. This leads to the information rent term that is common in auction settings. Second, probability mass of $f_{(2)}\left(x|x_S\right)$ is transferred from sale at reserve to sale at auction because the number of participating bidders changes from 1 to 2. For those sales, the price paid switches from the reserve price $w(x, \hat{x}_S)$ to the auction price $v(x, \hat{x}_S)$. This leads to the *linkage effect* term.

Some remarks on the linkage effect are in order. First, it is specific to the interdependent-value setup. If a buyer's valuation does not depend on other buyers' signals, then $v(x, \hat{x}_S) =$

$w(x, \hat{x}_S)$. That is, auction price is equal to reserve price, and thus linkage effect is nil.

If values are strictly interdependent, however, then the auction price is higher on the margin than the reserve price, i.e., $v(x, \hat{x}_S) > w(x, \hat{x}_S)$. The inequality compares two scenarios: one is when the marginal buyer wins over another participating bidder who would drop out had the reserve price been slightly higher; and the other is the marginal participant wins when no one else presents in the auctions (with the expected value being $w(x, \hat{x}_S)$). The inequality simply says the winner's curse is more severe in the latter case. When competing with rivals, the bidder incorporates the higher signal of the other active bidders and updates the expected value of the object upwards to $v(x, \hat{x}_S)$. This information channel across buyers causes the auction price to go above the reserve price, and thus leads to a positive linkage effect.

Second, it is worth noting that the linkage effect term has an opposite sign on the auctioneer's virtual profit than the information rent term. Since high types can always pretend to be low types, selling to a low type entails giving information rent to all higher types. Such consideration may cause the monopolist to withhold the object sometimes even when the buyer's expected value is higher than the seller's cost and thus selling is efficient. This is the familiar reason why monopoly results in too little sale. Linkage effect, on the other hand, has an opposite effect. A lower reserve price allows relatively good information to be revealed since not participating is always bad news. Such good news alleviates winner's curse and increases the price the winner pays. This cross-buyer information channel incentivises the seller to lower reserve price to encourage participation. As we will show later, linkage effect might overturn the conclusion that monopoly results in too little sale.

3 Separating Equilibrium

We study existence and characterization of a separating equilibrium. We follow the literature and consider equilibrium strategy $m(x_S)$ that determines the marginal buyer type, with the equilibrium price $w(m(x_S), x_S)$. Deviating to a price different from the prescribed equilibrium price is equivalent to signaling a type $\hat{x}_S$ different from the true type of x_S. The buyer will respond accordingly, with the marginal participation type becoming $m(\hat{x}_S)$. If $m(\cdot)$ is a separating equilibrium strategy, the seller must weakly prefer to signal her true type.

In our setup, the seller faces a trade-off between signaling a higher type and implementing a higher probability of sale. In other words, if the seller wants a higher probability of sale, by implementing a *lower* marginal buyer type x, she has to compromise with a lower belief and hence a lower sale price.

In order to take a closer look at this trade-off, we consider the preference of a type x_S seller in the $(x, \hat{x}_S)$ domain. She faces indifference curves defined by $\pi(x, x_S, \hat{x}_S) = constant$, with the slope

$$\frac{d\hat{x}_S}{dx} = -\frac{\partial\pi/\partial x}{\partial\pi/\partial\hat{x}_S} = \frac{J(x, x_S, \hat{x}_S) f_{(1)}(x|x_S)}{\partial\pi/\partial\hat{x}_S} \tag{5}$$

where the second equality follows from (3). The slope, often referred to as the Marginal Rate of Substitution, reflects the relative importance between the two instruments a seller has in order to change her profits: the marginal buyer type and buyers' belief about her.

We further denote

$$K(x, x_S, \hat{x}_S) \equiv \frac{\partial\pi/\partial\hat{x}_S}{f_{(1)}(x|x_S)} = \frac{\partial w(x, \hat{x}_S)}{\partial\hat{x}_S} \frac{F_{(2)}(x|x_S) - F_{(1)}(x|x_S)}{f_{(1)}(x|x_S)} + \int_x^1 \frac{\partial v(\tilde{x}, \hat{x}_S)}{\partial\hat{x}_S} \frac{f_{(2)}(\tilde{x}|x_S)}{f_{(1)}(x|x_S)} d\tilde{x}. \tag{6}$$

Then, the slope of the seller's indifference curve can be represented by

$$\frac{d\hat{x}_S}{dx} = \frac{J(x, x_S, \hat{x}_S)}{K(x, x_S, \hat{x}_S)}.$$

Borrowing analogy from Krishna (2002), suppose there is only one buyer, whose value is drawn from distribution F. Then, to sell with probability q at a take-it-or-leave-it offer, the marginal buyer type must be $x = F^{-1}(1-q)$, and thus price must be $w\left(F^{-1}(1-q), \hat{x}_S\right)$ where $\hat{x}_S$ is buyers' belief about the seller's type. Assumption 2 guarantees that the demand curve the seller faces, defined as $P = w\left(F^{-1}(1-q), \hat{x}_S\right)$ as a function of probability of sale q, is downward sloping. Krishna(2002) points out that virtual profit $J(\cdot)$ is simply the seller's marginal profit, which can be represented by the area of the rectangle $q_0 q_1 BD$ minus the area of $p_0 p_1 AB$ in figure 1. In our environment, the seller can affect her profits not only through probability of sale q, but also by manipulating buyers' belief about her type. The latter amounts to shifting the entire demand curve $w\left(F^{-1}(q), \hat{x}_S\right)$ by changing $\hat{x}_S$, resulting in $K(\cdot)$ being its marginal impact on profits after adjusting for the $q - x$ conversion. So $K(\cdot)$ is proportional to the area $p_0 p' CA$ in figure 1. Thus it is equal to 0 if probability of sale is $q = 0$.

J/K measures the increase in belief necessary to compensate for an increase in marginal buyer type. If we think of marginal buyer type x as the signal the seller can choose, like years

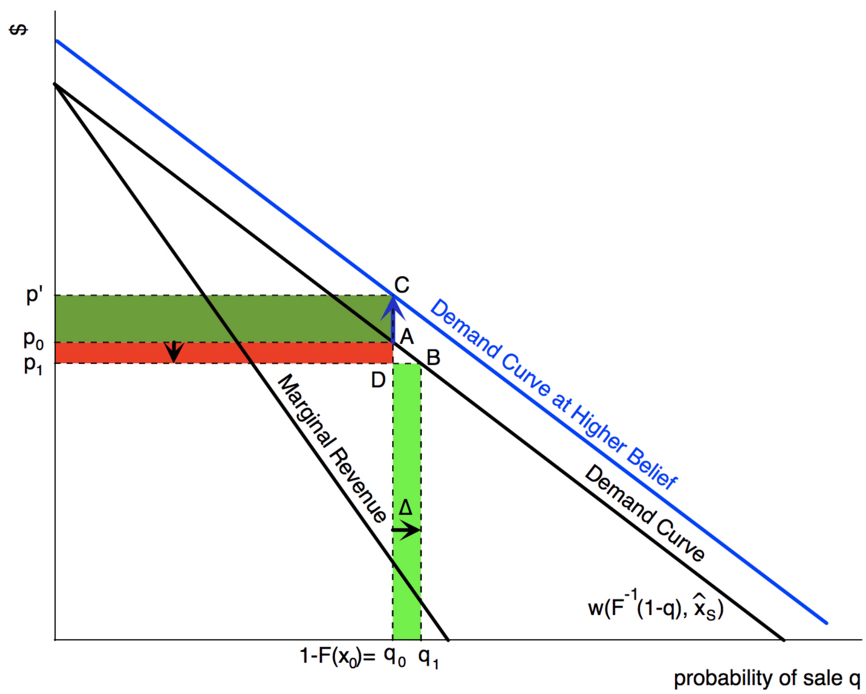


Figure 1: Effect on Profits of the Seller's Two Instruments: Probability of Sale q and Buyers' belief $\hat{x}_S$

of education an employee can choose in the classic job market signalling model, then J/K is the seller's marginal cost of signaling (measured in terms of buyers' beliefs). Then, a necessary condition for existence of separating equilibria is the following *Spence-Mirrlees* single-crossing condition – higher type has lower marginal costs of signalling.

Assumption 3 (Spence-Mirrlees)**.** *The ratio*

$$\frac{J(x, x_S, \hat{x}_S)}{K(x, x_S, \hat{x}_S)}$$

is decreasing in x_S *for all* $(x, \hat{x}_S)$*.*

3.1 Uniqueness

We now show that, in a separating equilibrium, there is "no distortion" at the bottom. To discuss distortion introduced by signaling concern, we need to consider the case of *complete* information, i.e. when the seller's type x_S is common knowledge. In this case, the seller will set the price $w(m^*(x_S), x_S)$, where $m^*(x_S)$ solves the first-order condition

$$\frac{\partial \pi(m^*(x_S), x_S, x_S)}{\partial x} = 0 \implies J\left(m^*(x_S), x_S, x_S\right) = 0.$$

We make the following assumption so the optimal complete information cutoff type is interior, and the solution to the above first-order condition is unique and globally optimal.

Assumption 4. *For all* $x_S \in [0,1]$*,* $J(0, x_S, x_S) < 0$ *and* $J(1, x_S, x_S) > 0$*. Moreover, for any* $(x, x_S) \in [0,1]^2$*,*

$$\frac{\partial J\left(x, x_S, x_S\right)}{\partial x} > 0. \tag{7}$$

Write $\underline{x} = m(0)$ and $\overline{x} = m(1)$. So $\underline{x}$ and $\overline{x}$ are the equilibrium marginal buyer type chosen by the lowest and the highest seller type respectively. We now show that, in a separating equilibrium, the lowest seller type chooses the marginal buyer type that is his complete-information-optimum. So there is "no distortion" at the bottom.

Lemma 1 (Initial Condition)**.** *There is a unique lowest marginal buyer type compatible with a separating equilibrium, and it is equal to type* 0 *seller's complete information optimal cutoff type:*

$$\underline{x} = m^*(0).$$

Proof. For type 0 seller, by definition of $\underline{x}$ being her equilibrium cutoff type, choosing the cutoff $\underline{x}$ and revealing her true type of 0 must be weakly better than choosing any other cutoff type x and the corresponding equilibrium belief $s(x)$. So

$$\pi\left(\underline{x},0,0\right)\geq\pi\left(x,0,s\left(x\right)\right)\geq\pi\left(x,0,0\right)forallx\in\left[0,1\right],$$

where the second inequality holds because profits are increasing in buyers' belief which is the third argument to π. The conclusion then follows because $m^*(0)$ is type 0 seller's optimal cutoff type if her type is common knowledge. □

We show in the next lemma that a separating equilibrium strategy, if it exists, must be differentiable. This is an implication of the incentive compatibility condition and the Spence-Mirrlees Assumption 3. Moreover, the slope of the equilibrium path in the $(x,\hat{x}_S)$ domain at point $(m(x_S),x_S)$ must be equal to the slope of type x_S seller's indifference curve.

Lemma 2. *A separating equilibrium* $m(x_S):[0,1]\rightarrow[0,1]$ *must be differentiable and satisfies*

$$m'\left(x_S\right)=-\frac{\partial\pi/\partial\hat{x}_S}{\partial\pi/\partial x} \tag{8}$$

for all $x_S\in[0,1]$.

Proof. In the Appendix. □

It turns out to be more convenient to state the resulting differential equation in terms of the *inverse* strategy

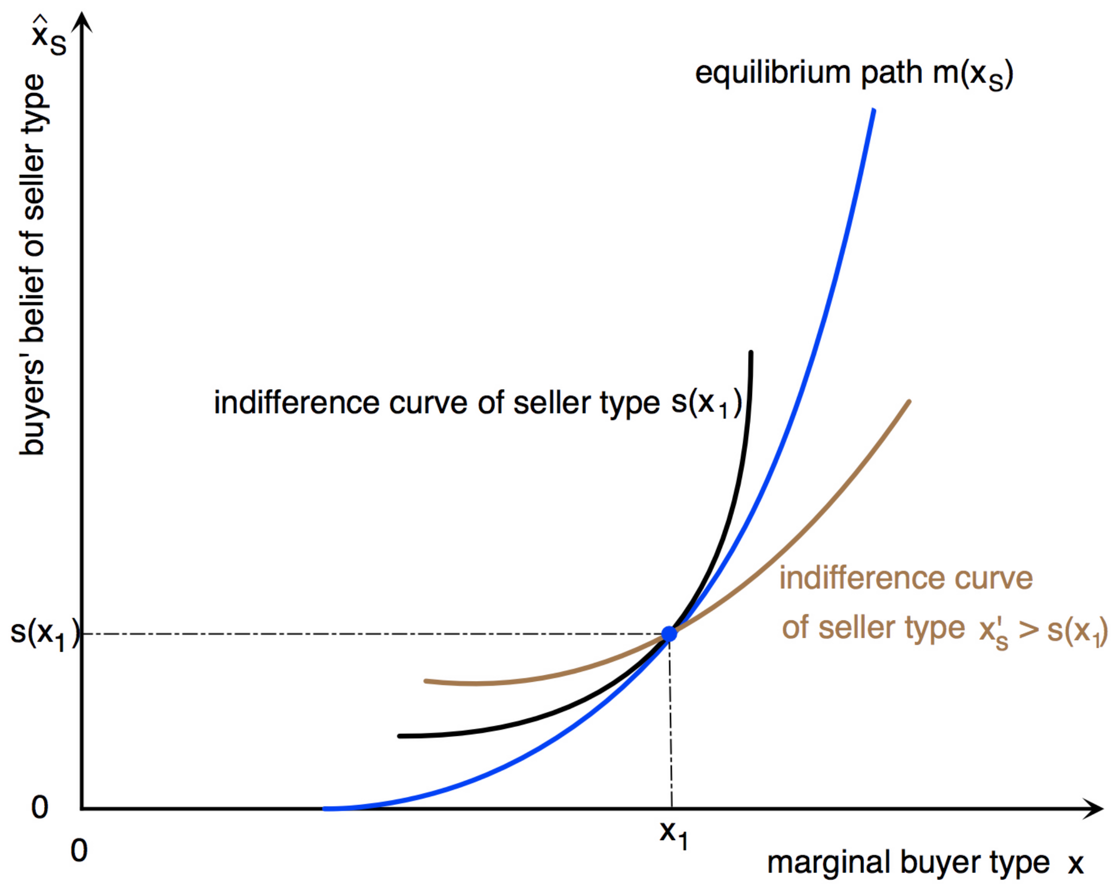
$$s(\cdot)\equiv m^{-1}(\cdot).$$

The inverse strategy $s(\cdot)$ is defined on the domain $[\underline{x},\overline{x}]$, where $\underline{x}=m(0)$ and $\overline{x}=m(1)$ are the lowest and highest marginal buyer types respectively. $s(\cdot)$ is continuously differentiable at any point of its domain $[\underline{x},\overline{x}]$. From equations (3), (6) and (8), we get the differential equation that our separating equilibrium must satisfy[6]

$$s'(x)=\frac{J(x,s(x),s(x))}{K(x,s(x),s(x))},\qquad s(\underline{x})=0, \tag{9}$$

Recall that $\frac{J(x,x_S,\hat{x}_S)}{K(x,x_S,\hat{x}_S)}$ is the slope of type x_S seller's indifference curve at $(x,\hat{x}_S)$. So, as illustrated in figure 3.1, the slope of an equilibrium path at $(x_1,s(x_1))$ must be equal to the

[6] This is in parallel to equation (5) in Lamy (2010).



slope of the seller's indifference curve at the same point when the seller's true type is equal to the buyers' belief $s(x_1)$. Assumption 3 says that higher seller types ($x'_S > s(x_1)$) have flatter indifference curves.

In our model, the seller signals through lower probability of trade. However, $K(x, x_S, \hat{x}_S)$ goes to 0 when $x \to 1$, and thus the right-hand-side of (9) is not well-defined at $x = 1$. This would not be a problem if one ensured that interior signals are used by all seller types. For example, all sellers prefer pooling with the lowest type to separating via the highest signal. (See Riley (1979).) In our environment, it is desirable to allow for situations where a high type seller may have to suffer a loss if he can not separate from the lowest type, as his own use value is too high. We get around the problem by solving for the trajectory of a vector field consistent with the differential equation (9) instead of solving it directly.

Lemma 3. *There exists a unique solution* $s : [\underline{x}, \overline{x}] \to [0, 1]$ *of the differential equation* (9)

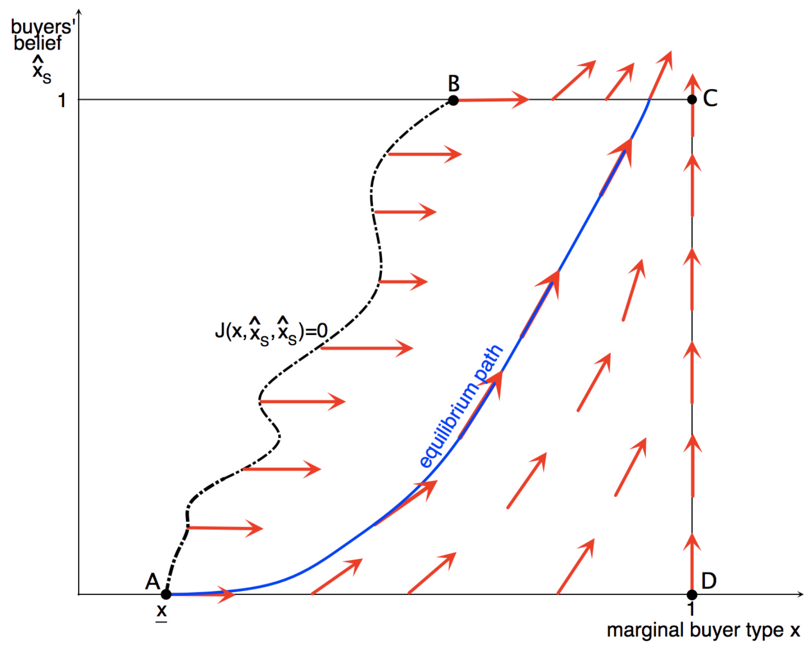


Figure 2: The arrows represent the vector field defined by (11) and (10). The dotted line is the curve defined by $J(x, \hat{x}_S, \hat{x}_S) = 0$. The blue solid line is the equilibrium path.

satisfying the initial condition $s(\underline{x}) = 0$*, and it is strictly increasing on* $[\underline{x}, \overline{x}]$ *with* $s'(x) > 0$ *for* $x \in (\underline{x}, \overline{x})$*.*

Proof. In order to facilitate the analysis, we show existence and uniqueness of the solution to the system of two autonomous differential equations:

$$\tilde{s}'(t) = J(\tilde{m}, \tilde{s}, \tilde{s}), \tag{10}$$

$$\tilde{m}'(t) = K(\tilde{m}, \tilde{s}, \tilde{s}), \tag{11}$$

where $\tilde{s}(t), \tilde{m}(t)$ are the seller and buyer types on the separating equilibrium curve, parameterized by $t \in R_+$, with the initial conditions $\tilde{s}(0) = 0$ and $\tilde{m}(0) = \underline{x}$. If there exists a unique solution to the above system, increasing and satisfying $\tilde{s}(T) = 1$ and $\tilde{m}(T) \equiv \overline{x} < 1$ for some $T > 0$, then there will also exist a unique solution $s = \tilde{s} \circ \tilde{m}^{-1} : [\underline{x}, \overline{x}] \to [0, 1]$ of the differential equation (9). This is because

$$s'(x) = \frac{\tilde{s}'(t)}{\tilde{m}'(t)}\Big|_{t=\tilde{m}^{-1}(x)} = \frac{J(x, s(x), s(x))}{K(x, s(x), s(x))}.$$

Note that the system (10) and (11) of two well-behaved differential equations for $\tilde{s}(\cdot), \tilde{m}(\cdot)$ avoids the "small denominator" problem in the original equation (9) since K approaches 0 as x approaches 1. The phase portrait of this system is shown in Figure 2. Let

$$M \equiv \{(x, x_S) \in [\underline{x}, 1] \times [0,1] : J(x, x_S, x_S) \geq 0\}$$

be the domain in the interior of which the equilibrium candidate is strictly increasing, represented by the curvilinear region ABCD in Figure 2. By Assumption 1, $w(x, x_S), v(x, x_S), u(x_S, x)$ are twice continuously differentiable on $[0,1]^2$. By lemma 5 in the Appendix, $J(\tilde{m}, \tilde{s}, \tilde{s})$ and $K(\tilde{m}, \tilde{s}, \tilde{s})$ are continuously differentiable with respect to both $\tilde{m}$ and $\tilde{s}$ for $(\tilde{m}, \tilde{s}) \in [\underline{x}, 1] \times [0,1]$. By a standard theorem for differential equations, there is a unique solution $(\tilde{m}, \tilde{s})$ passing through the initial condition $(\tilde{m}(0), \tilde{s}(0)) = (\underline{x}, 0)$.[7] The solution curve must reach the boundary of M for some $T > 0$. By the way of contradiction, suppose the solution is contained in M $\forall t \geq 0$. Since $(J(x, x_S, x_S), K(x, x_S, x_S)) \neq (0,0)$ on M, there are no critical points in M of the vector field defined by the r.h.s. of (10) and (11). Then the Poincare-Bendixson theorem implies that the solution must approach a closed orbit as $t \to \infty$, which must also be contained in M.[8] But this is impossible because such a closed orbit must contain a critical point inside, but there are no critical points in M.[9] This establishes the required contradiction. Because the vector field defined by the system is parallel to the horizontal axis on the J-curve, to the vertical axis on the right side of the square, where $x = 1$, and points northeast on the segment AD on the bottom horizontal side of the square, and because the initial condition $(\underline{x}, 0)$ is on the boundary of M, the solution curve cannot leave M through these boundaries. Therefore, the solution must leave the M region through the segment BC on the upper horizontal side of the square, so that $\tilde{s}(T) = 1$, while $\tilde{m}(T) = \overline{x} < 1$, and $(\tilde{m}(t), \tilde{s}(t))$ belongs to M for all $t \in [0, T]$. Moreover, since $J(\tilde{m}, \tilde{s}, \tilde{s}) > 0$ and $K(\tilde{m}, \tilde{s}, \tilde{s}) > 0$ for $(\tilde{m}, \tilde{s})$ in the interior of M, $(\tilde{m}(t), \tilde{s}(t))$ belongs to the interior of M and $\tilde{s}'(t) > 0$, $\tilde{m}'(t) > 0$ for $t \in (0, T)$. This implies that $s = \tilde{s} \circ \tilde{m}^{-1}$ satisfies $s'(x) > 0$ for $x \in (\underline{x}, \overline{x})$, and therefore $s(\cdot)$ is increasing on $[\underline{x}, \overline{x}]$. □

The unique solution of (9) gives us a unique *candidate* for a separating equilibrium. In order to show that it in fact defines a separating equilibrium, we need to verify that there are

[7] See e.g. the second theorem in Section 2.2 of Anosov and Arnold (1994).

[8] See e.g. Theorem 1.8.1 on p44 in Guckenheimer and Holmes (1983).

[9] See Corollary 1.8.5 on p51 in Guckenheimer and Holmes (1983.

no incentives to deviate, either to within-equilibrium or out-of-equilibrium prices.

Our main result is the proposition below that shows existence and uniqueness of the separating equilibrium under the maintained assumptions. The proof shows that deviations from the equilibrium candidate are not profitable. The Spence-Mirrlees condition rules out within-equilibrium deviations. But out of equilibrium beliefs need also be addressed. The highest and lowest equilibrium reserve prices are given by

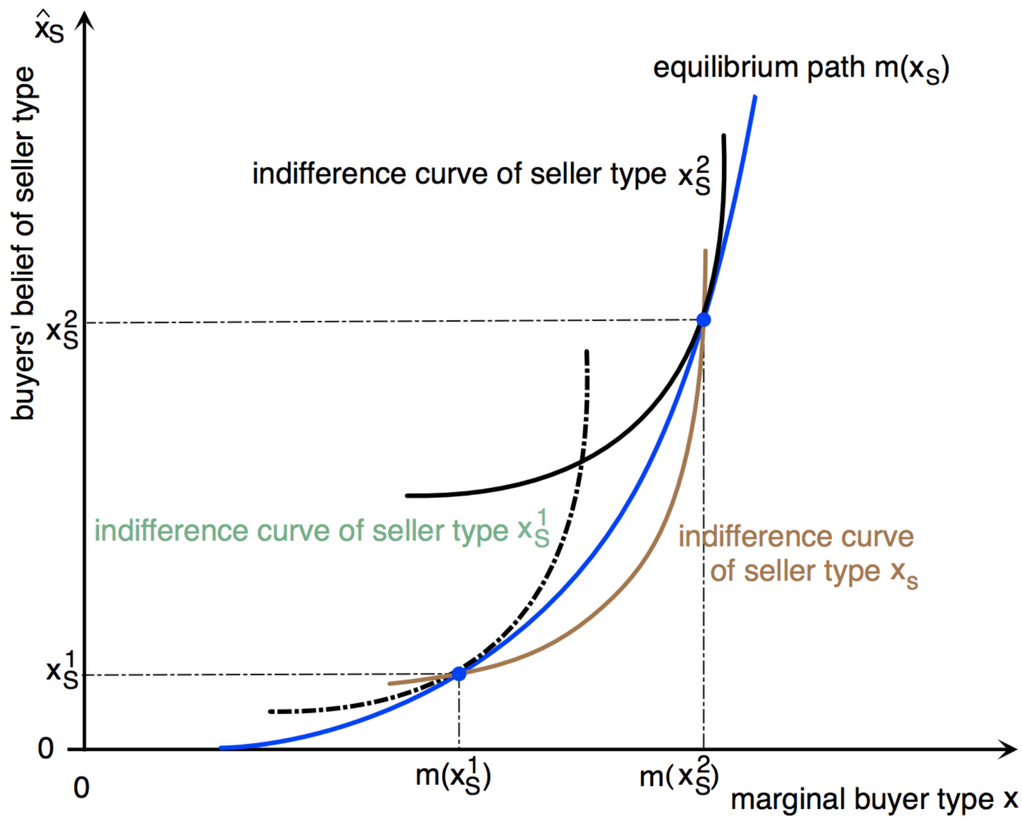
$$\underline{p} = w(\underline{x}, 0), \qquad \overline{p} = w(\overline{x}, 1),$$

where $\underline{x}$ and $\overline{x}$ are the lowest and the highest equilibrium marginal buyer type respectively (see Figure **??**). The equilibrium is supported by the most pessimistic beliefs for prices below $\underline{p}$, and arbitrary beliefs for prices above $\overline{p}$.

Consider seller type x_S. If she pretends to be type $x_S^2 > x_S$, then her marginal buyer type will be $m\left(x_S^2\right)$, as shown in figure 3.1. By construction of the candidate equilibrium path (blue curve in figure 3.1), the slope of the equilibrium path at $\left(m\left(x_S^2\right), x_S^2\right)$ must be equal to the slope of type x_S^2 seller's indifference curve at that point. By Spence-Mirrlees condition of decreasing marginal cost of signalling, and recall that the slope of the seller's indifference curve reflects the seller's marginal cost of signalling, type x_S^2 seller's indifference curve passing through $\left(m\left(x_S^2\right), x_S^2\right)$ will be flatter than type x_S seller's indifference curve. Since the seller's profits increase strictly with buyers' belief given that $m\left(x_S^2\right) \leq \overline{x} < 1$, type x_S seller's profits along the equilibrium path are strictly decreasing in a neighborhood around $\left(m\left(x_S^2\right), x_S^2\right)$ for $\hat{x}_S > x_S$. Similar reasoning shows that type x_S seller's profits along the equilibrium path are strictly increasing in a neighborhood around $(m(\hat{x}_S), \hat{x}_S)$ for $\hat{x}_S < x_S$. It follows that the equilibrium combination $(m(x_S), x_S)$ maximizes type x_S seller's profits among $(m(\hat{x}_S), \hat{x}_S)$ for all $\hat{x}_S \in [0, 1]$.

Suppose type x_S seller chooses marginal buyer type $x < \underline{x}$. Since $J(\underline{x}, 0, 0) = 0$, by Assumption 4, for all $x < \underline{x}$, $J(x, 0, 0) < 0$ and thus the slope of type 0 seller's indifference curve at $(x, 0)$ is negative. By Spence-Mirrlees signaling condition, the slope of type x_S seller's indifference curve at $(x, 0)$ is also negative, for all $x_S > 0$. So for all $x_S \geq 0$, profits from choosing marginal cutoff type $x < \underline{x}$ is increasing strictly in x.

Suppose type x_S seller chooses marginal buyer type $\hat{x} > \overline{x}$. We have shown that $(\overline{x}, s(\overline{x})) =$



$(\overline{x}, 1) \in M$. That is, $J(\overline{x}, 1, 1) \geq 0$. By Assumption 4, $J(x, 1, 1) > 0$ for all $x \in (\overline{x}, 1]$. So, for $x \in (\overline{x}, 1]$, the slope of type 1 seller's indifference curve at $(x, 1)$ is positive. By the Spence-Mirrles condition, hence the slope of type x_S seller's indifference curve at $(x, 1)$ is even larger and therefore positive, for any $x_S \in [0, 1]$. So the seller's profits from $(x, 1)$ is strictly decreasing in x for $x \in (\overline{x}, 1]$. So the seller prefers combination $(\overline{x}, 1)$ to $(\hat{x}, 1)$, thus prefers $(\overline{x}, 1)$ to $(\hat{x}, s(\hat{x}))$ since profits are increasing in buyers' belief.

Proposition 1. *Under Assumptions 1-4, there exists a unique separating equilibrium in increasing and differentiable strategies, characterized by the differential equation* (9) *with the initial condition* $s(\underline{x}) = 0$*, where* $\underline{x} = m^*(0)$ *as defined in the lemma 1. For reserve prices offers outside the equilibrium range* $[\underline{p}, \overline{p}]$*, this equilibrium is supported by out-of-equilibrium buyer beliefs as follows:*

- *For* $p < \underline{p}$*, the buyer believes that the price originated from the seller with the lowest signal,* $x_S = 0$*;*

- *For $p > \overline{p}$, the buyer's beliefs are unrestricted.*

Proof. Consider the seller of type x_S who deviates within the equilibrium and signals $\hat{x}_S$. Since $m(\hat{x}_S)$ is differentiable for $\hat{x}_S \in (0,1)$, it follows that $\pi(m(\hat{x}_S), x_S, \hat{x}_S)$ is differentiable for $\hat{x}_S \in (0,1)$ and therefore it is sufficient to only check the deviations $\hat{x}_S \in (0,1)$. For those deviations, the slope of $\pi(m(\hat{x}_S), x_S, \hat{x}_S)$ is

$$\begin{aligned}
\frac{d\pi(m(\hat{x}_S), x_S, \hat{x}_S)}{d\hat{x}_S} &= \Big(- m'(\hat{x}_S) J(m(\hat{x}_S), x_S, \hat{x}_S) + K(m(\hat{x}_S), x_S, \hat{x}_S) \Big) f_{(1)}(m(\hat{x}_S)|x_S) \\
&= \Big(\frac{1}{m'(\hat{x}_S)} - \frac{J(m(\hat{x}_S), x_S, \hat{x}_S)}{K(m(\hat{x}_S), x_S, \hat{x}_S)} \Big) K(m(\hat{x}_S), x_S, \hat{x}_S) m'(\hat{x}_S) f_{(1)}(m(\hat{x}_S)|x_S) \\
&= \Big(\frac{J(m(\hat{x}_S), \hat{x}_S, \hat{x}_S)}{K(m(\hat{x}_S), \hat{x}_S, \hat{x}_S)} - \frac{J(m(\hat{x}_S), x_S, \hat{x}_S)}{K(m(\hat{x}_S), x_S, \hat{x}_S)} \Big) K(m(\hat{x}_S), x_S, \hat{x}_S) m'(\hat{x}_S) f_{(1)}(m(\hat{x}_S)|x_S)
\end{aligned}$$

where we have used the fact that (9) implies

$$m'(\hat{x}_S) = \frac{K(m(\hat{x}_S), \hat{x}_S, \hat{x}_S)}{J(m(\hat{x}_S), \hat{x}_S, \hat{x}_S)} > 0, \qquad \hat{x}_S \in (0,1).$$

By the Spence-Mirrlees condition (Assumption 3), the expression in the parentheses is positive for $\hat{x}_S < x_S$, 0 for $\hat{x}_S = x_S$ and negative for $\hat{x}_S > x_S$, while $K(m(\hat{x}_S), x_S, \hat{x}_S) > 0$. Thus $\pi(m(\hat{x}_S), x_S, \hat{x}_S)$ is uniquely maximized at $\hat{x}_S = x_S$ and within-equilibrium path deviations are not profitable. We now show that out-of-equilibrium path deviations are not profitable either. Consider first a deviation to $p < \underline{p}$. As the beliefs following such a deviation are the most pessimistic, $\hat{x}_S = 0$, the effect of the seller's deviation will be only changing the marginal buyer type, and the slope of the expected profit for $x < \underline{x}$ is equal to

$$\begin{aligned}
\frac{\partial \pi(x, x_S, 0)}{\partial x} &= -J(x, x_S, 0) f_{(1)}(x|x_S) \\
&= -\frac{J(x, x_S, 0)}{K(x, x_S, 0)} K(x, x_S, 0) f_{(1)}(x|x_S) \\
&\geq -\frac{J(x, 0, 0)}{K(x, 0, 0)} K(x, x_S, 0) f_{(1)}(x|x_S) \\
&\geq -\frac{J(\underline{x}, 0, 0)}{K(x, 0, 0)} K(x, x_S, 0) f_{(1)}(x|x_S) \\
&= 0,
\end{aligned}$$

where the first inequality follows by the Spence-Mirrlees condition (Assumption 3), and the second inequality by the assumption that J is increasing in x (Assumption 4). Thus the slope of the expected profit at such a deviation is non-negative, showing that the deviation is not profitable. As for the deviation to a price $p > \overline{p}$ (equivalently, a deviation to $x > \overline{x}$), we first note that, since we have ruled out within-equilibrium deviations, it is not profitable to deviate to the highest in-equilibrium value $\overline{x}$:

$$\pi(m(x_S), x_S, x_S) \geq \pi(\overline{x}, x_S, 1) \geq \pi(\overline{x}, x_S, \hat{x}_S).$$

Next, the slope of the most optimistic profit $\pi(\overline{x}, x_S, 1)$ with respect to x for $x > \overline{x}$ following such a deviation is

$$\begin{aligned}
\frac{\partial \pi(\overline{x}, x_S, 1)}{\partial x} &= -J(\overline{x}, x_S, 1) f_{(1)}(x|x_S) \\
&= -\frac{J(\overline{x}, x_S, 1)}{K(\overline{x}, x_S, 1)} K(\overline{x}, x_S, 1) f_{(1)}(x|x_S) \\
&\leq -\frac{J(\overline{x}, 1, 1)}{K(\overline{x}, 1, 1)} K(\overline{x}, x_S, 1) f_{(1)}(x|x_S) \\
&\leq 0,
\end{aligned}$$

where the first inequality follows by the Spence-Mirrlees Assumption 3, and the second — because $J(m(1), 1, 1)/K(m(1), 1, 1) = s'(\overline{x}) \geq 0$ by Lemma 3.[10] Thus the slope of the most optimistic expected profit is always non-positive for $x > \overline{x}$, implying that it is not optimal to deviate there for any belief. □

3.2 Implications of linkage effect

3.2.1 Spence-Mirrlees condition

The Spence-Mirrlees single crossing condition plays an important role in guaranteeing existence and uniqueness of a separating equilibrium in our model. Though it is a standard condition for the theory of signalling games, it is desirable to connect this connection more directly to auction theory, and explore stronger but arguably simpler conditions that would imply Spence-Mirrlees condition in our model.

The Spence-Mirrlees condition requires the slope of the seller's indifference curve in the $x - \hat{x}_S$ domain, J/K, to decrease with the seller's type. We first see that K increases with the seller's type. This is a direct implication of the following observation.

[10] Lemma 3 shows $s'(x) > 0$ for $x \in (\underline{x}, \overline{x})$, and therefore $s'(\overline{x}) \geq 0$ by the continuity of $s(\cdot)$.

Lemma 4. $\frac{F_{(2)}(x|x_S)-F_{(1)}(x|x_S)}{f_{(1)}(x|x_S)}$ *increases in* x_S. *Moreover, for any* $x' \geq x$, $\frac{f_{(2)}(x'|x_S)}{f_{(1)}(x|x_S)}$ *also increases in* x_S.

This observation is a result of affiliation between seller's and buyers' signals. The proof is provided in the Appendix.

Intuitively, since K measures the impact on profits from an increase in belief $\hat{x}_S$, and belief affects profits only through the price the winner pays conditional on sale, K would increase with probability of sale. Since sale takes place if highest buyer type is above the marginal type, affiliation implies that higher seller types are more optimistic about probability of sale.

J/K measures the relative importance of the seller's two instruments to increase profits: selling with higher probability and eliciting higher belief. Since higher seller types care more about eliciting higher belief, i.e. K is increasing in seller type, Spence-Mirrlees condition holds if higher types care less about selling with higher probability. Following **?** and employing the probability of sale $q = 1 - F_{(1)}(x)$ as the seller's decision variable, the seller's marginal profits are equal to $J(x, x_S, \hat{x}_S)$. The following assumption then says that higher seller type has lower marginal profits and thus care less about making a sale and thus are more willing to set a high reserve price.

Assumption 5. *The virtual profit function* $J(x, x_S, \hat{x}_S)$ *is decreasing in* x_S *for any* $(x, \hat{x}_S)$.

When will marginal profits be decreasing in seller type? That is, when is Assumption 5 satisfied? Assumption 2 implies that higher seller types have higher marginal costs. The first part of lemma 4 implies that the size of information rent is also increasing in seller type. This is intuitive. Including a particular type means giving information rent to all higher types because they can now pretend to be this low type. By affiliation, higher seller types believe in a higher likelihood ratio of buyer type being above a cutoff to it being equal to a cutoff. So higher seller types believe that they have to pay a higher information rent to include a given buyer type. Thus, both the cost term and information term imply that higher seller types are less eager to sell.

However, the positive linkage effect is increasing in seller type x_S. If a slight reduction in reserve price increases the number of participants from 1 to 2, the auction becomes contested and the price the winner pays increases from the reserve price to the auction. This is because participation of a second bidder reveals good news: the second highest signal is above rather than below the marginal type. And the assumption of interdependent value translates this good news into higher willingness-to-pay and thus higher price. So linkage effect measures the monopolist's incentives to include a type due to this cross-buyer information channel. By affiliation, the higher the seller type x_S is, the more favorable he thinks buyers' signals are. So the more likely he thinks that a small reserve price reduction will lead to participation of a second bidder. So this cross-buyer information channel is more important for higher seller types, making higher types more eager to sell.

Therefore, seller of higher types has lower marginal profits and thus less eager to sell if the interaction between seller type and linkage effect is sufficiently weak. There are two extreme cases. The first is when the buyers' values are *conditionally independent* given x_S, in which case the linkage effect disappears. The second is when seller's and buyers' signals are not affiliated, so seller's belief about distribution of buyers' signals is independent of her own type x_S. In that case, neither information rent or linkage effect changes with the seller's type. Thus marginal profits change with seller type only via marginal costs, and thus are decreasing in seller type.

In general, the interaction between seller type and linkage effect is sufficiently weak if the combination of interdependent value (contributing to the gap $v - w$ between price at auction and price at reserve) and affiliated signals (contributing to the different beliefs held by seller of different types about the relative importance of the cross-buyer information channel) is sufficiently small. The Proposition below formalizes this idea and gives an explicit bound: the rate at which the linkage effect term changes with seller type has to be smaller than rate at which marginal costs change with seller type.

Proposition 2. *Assumption 5 implies that* $\frac{J(x,x_S,\hat{x}_S)}{K(x,x_S,\hat{x}_S)}$ *decreases in* x_S. *Moreover, Assumption 5 is satisfied, if*

$$(v(x,\hat{x}_S) - w(x,\hat{x}_S)) \frac{\partial}{\partial x_S} \left(\frac{f_{(2)}(x|x_S)}{f_{(1)}(x|x_S)} \right) < \frac{\partial u_S(x_S,x)}{\partial x_S}. \tag{12}$$

On the other hand, if there is strong interdependence between buyer values and strong affiliation between buyers' and the seller's signals, linkage effect is strong and seller of higher types may deem this cross-buyer information channel so much more important than seller of lower types that higher types may be more eager to sell than lower types, thereby destroying existence of a separating equilibrium. We provide such an example later.

3.2.2 Welfare Implication of Signaling

Is the ability to signal a blessing from the social perspective?

Suppose the seller's type is public information. Therefore, there is no need to signal, and the seller will simply choose the complete information optimal reserve. If buyers' values are independent, then the linkage effect term is nil. Then, the seller will set a reserve price above his marginal cost because selling to a low buyer type involves giving information rent to all higher types. Therefore, the marginal type must have interim willingness-to-pay, $w(x, x_S)$, strictly above the seller's marginal cost to cover the loss in profits due to information rent. This is the familiar theme that quantity sold is inefficiently low under monopoly. In this case, signaling concern exacerbates the problem. To separate from lower types and signal his true type, the seller of each type except for the lowest sets a reserve even higher than the complete-information-optimum. Therefore, the opportunity to signal has negative welfare implication.

However, when buyers' values are interdependent, linkage effect is present. Presence of a competing bidder can boost the price the winner pays because her participation reveals positive information. Linkage effect incentivises the seller to encourage participation by setting a low reserve price, in contrast to the effect of information rent. If linkage effect is strong, it can overturn the conventional wisdom about monopoly and cause the monopolist to set a reserve below his marginal cost. In this situation, quantity sold under monopoly is inefficiently HIGH. The monopolist sometimes sells the good to a low type below cost in the hope such a type's participation reveals good information to the winner and increases the price the winner pays.

But if quantity sold is inefficiently high under complete information, signaling concern then corrects this inefficiency. It is then not clear whether the opportunity to signal increases or decreases social welfare. Therefore, taking linkage effect into consideration not only affects our understanding about the *direction* of inefficiency caused by monopoly, but also alters the welfare implication of existence of signaling channel.

Appendix

Proof of lemma 2. Recall that

$$\begin{aligned}\frac{\partial\pi\left(x,x_S,\hat{x}_S\right)}{\partial x} &= J\left(x,x_S,\hat{x}_S\right)f_{(1)}\left(x|x_S\right)\\ \frac{\partial\pi\left(x,x_S,\hat{x}_S\right)}{\partial \hat{x}_S} &= \kappa\left(x,x_S,\hat{x}_S\right)f_{(1)}\left(x|x_S\right).\end{aligned}$$

Our environment satisfies assumption (1), (4), (5), (6) and (7) in Mailath (1987). More specifically, (1) says π is C^2 on $[0,1]^3$ and is implied by our assumption 1 and 2, (6) is implied by our Initial Condition lemma 1, (7) by the Spence-Mirreless assumption 3, (4) and (5) by the following claim and because $\pi \in C^2\left([0,1]^3\right)$.

Claim 1. *$\frac{\partial\pi(x,x_S,x_S)}{\partial x} = 0$ at $x = m^*\left(x_S\right)$, and $\frac{\partial\pi(x,x_S,x_S)}{\partial x} > 0$ if $x < m^*\left(x_S\right)$ and $\frac{\partial\pi(x,x_S,x_S)}{\partial x} < 0$ if $x > m^*\left(x_S\right)$.*

This claim holds because of our assumption **??** and 4 and because $f_{(1)}\left(x|x_S\right) > 0$ on $[0,1]^2$ by our assumption 1. Assumption (2) in Mailath (1987) requires seller's payoff to be strictly increasing in buyer's belief on the entire domain. Inspecting (**??**) and by our assumption 2, we obtain the following relationship for our environment.

Claim 2. *$\partial\pi/\partial\hat{x}_S \geq 0$ on $\left(x,x_S,\hat{x}_S\right) \in [0,1]^3$, with strict inequality on $[0,1)\times[0,1]^2$ and equality if $x = 1$.*

Though our environment violates assumption (2) in Mailath (1987), their proof relies only on the following observation.

Claim 3. *$\frac{\partial\pi(m^*(x_S),x_S,x_S)}{\partial\hat{x}_S} \neq 0$ for all $x_S \in [0,1]$.*

To see this, we first observe that, by assumption **??**, 4 and because $\pi \in C^2\left([0,1]^3\right)$, $m^*\left(x_S\right)$ solves $J\left(x,x_S,x_S\right) = 0$ and exhibit the following properties.

Claim 4. *$m^*\left(x_S\right) \in (0,1)$ for all $x_S \in [0,1]$, and has uniformly bounded derivates on $[0,1]$.*

Note that Claim 3 then follows from claim 2 and Claim 4. Let $m\left(x_S\right)$ be a separating equilibrium strategy that determines the marginal buyer type. Similar to Mailath (1987, Appendix, Proposition 2), if m is continuous at $x'_S \in [0,1]$, then using Taylor expansion and the equilibrium property that type x''_S prefers signal and belief pair $(x, \hat{x}_S) = (m(x''_S), x''_S)$ to $(m(x'_S), x'_S)$ and type x'_S has the reverse preference, we get

$$0 = \frac{\partial \pi (m(x'_S), x'_S, x'_S)}{\partial \hat{x}_S} + \lim_{x''_S \to x'_S} \frac{m(x''_S) - m(x'_S)}{x''_S - x'_S} \frac{\partial \pi (m(x'_S), x'_S, x'_S)}{\partial x}. \tag{13}$$

It follows that m is differentiable at x'_S and (8) holds if $\frac{\partial \pi(m(x'_S), x'_S, x'_S)}{\partial x} \neq 0$, which is the case if $m(x'_S) \neq m^*(x'_S)$, by claim 1. Next, monotonicity of $m(.)$ follows from the same arguments as in Riley (1979, lemma 2), which rely only on the Spence-Mirrless assumption 3 and the assumption that $\pi(x, x_S, \hat{x}_S)$ is weakly increasing in belief $\hat{x}_S$.

Claim 5. *$m(x_S)$ is weakly increasing in x_S on $[0,1]$.*

We can then show that the, in equilibrium, signaling incentive reduces probability of trade from the full-information optimum.

Claim 6. *$m(x'_S) \geq m^*(x'_S)$ if m is continuous at x'_S.*

Suppose to the contrary. Then, $\frac{\partial \pi(m(x'_S), x'_S, x'_S)}{\partial x} > 0$ by assumption 1 and $\frac{\partial \pi(m(x'_S), x'_S, x'_S)}{\partial \hat{x}_S} > 0$ by claim 2. By (13), $m'(x'_S) < 0$, contradiction to lemma 5. Following Mailath (1987, Appendix, Proposition 3), we show that, if $m(.)$ is continuous on an open interval $I \in [0,1]$, then $m(x_S) \neq m^*(x_S)$ for all $x_S \in I$. Suppose to the contrary. First, suppose $m(x_S) = m^*(x_S)$ on an open subinterval $E \subset I$. Then, for all $x_S \in I$, (1) $m(.)$ has uniformly bounded derivatives by claim 4, and (2) $\frac{\partial \pi(m(x_S), x_S, x_S)}{\partial x} = 0$. By (13), $\frac{\partial \pi(m(x_S), x_S, x_S)}{\partial \hat{x}_S} = 0$ for all $x_S \in I$, contradiction to claim 3. Therefore, we can w.l.o.g. suppose $m(x_S) \neq m^*(x_S)$ for all $x_S \in I$ but $m(x'_S) = m^*(x'_S)$ for some $x'_S \in I$. For $x_S \in I \backslash \{x'_S\}$, $\frac{\partial \pi(m(x_S), x_S, x_S)}{\partial x} < 0$ by lemma 6 and claim 1. For all $x_S < x'_S$, $m(x_S) \leq m(x'_S) = m^*(x'_S) \in (0,1)$ by lemma 5 and 4, and thus $\frac{\partial \pi(m(x_S), x_S, x_S)}{\partial \hat{x}_S}$ is bounded below by a positive number on $x_S \in [0, x'_S]$ by 2 and continuity. By (13), for $x_S \in I$ where $x_S < x'_S$, $m'(x_S)$ exists and goes to ∞ as $x_S \to (x'_S)^-$. Since $m^*(.)$ has uniformly bounded derivatives on $[0,1]$, $m(x_S) < m^*(x_S)$ for all $x_S < x'_S$ sufficiently close to x'_S, contradictioon to lemma 6. Suppose m is discontinuous at $x'_S \in (0,1)$. Because m is weakly increasing, the left limit and right limit exists, denoted by $m_-(x'_S)$ and $m_+(x'_S)$ respectively. Because

$$\pi\left(m\left(x''_S\right), x''_S, x''_S\right) - \pi\left(m\left(x'_S\right), x''_S, x'_S\right) \geq 0 \geq \pi\left(m\left(x''_S\right), x'_S, x''_S\right) - \pi\left(m\left(x'_S\right), x'_S, x'_S\right),$$

by taking left and right limits separately and by continuity of π, we get

$$\pi\left(m_-\left(x_S'\right), x_S', x_S'\right) - \pi\left(m\left(x_S'\right), x_S', x_S'\right) \geq 0 \geq \pi\left(m_-\left(x_S'\right), x_S', x_S'\right) - \pi\left(m\left(x_S'\right), x_S', x_S'\right)$$

and

$$\pi\left(m_+\left(x_S'\right), x_S', x_S'\right) - \pi\left(m\left(x_S'\right), x_S', x_S'\right) \geq 0 \geq \pi\left(m_+\left(x_S'\right), x_S', x_S'\right) - \pi\left(m\left(x_S'\right), x_S', x_S'\right)$$

Thus

$$\pi\left(m_-\left(x_S'\right), x_S', x_S'\right) = \pi\left(m_+\left(x_S'\right), x_S', x_S'\right).$$

Because π is strictly quasi-concave in marginal buyer type x and m is weakly increasing, $m^*\left(x_S'\right) \in \left(m_-\left(x_S'\right), m_+\left(x_S'\right)\right)$. Then, there exists $\delta > 0$ such that for all $x_S'' \in \left(x_S' - \delta, x_S'\right)$,

$$m^*\left(x_S''\right) > m_-\left(x_S'\right) > m\left(x_S''\right) \tag{14}$$

where the first inequality comes from continuity of m^* and the second from lemma 5. By monotonicity of m on $[0,1]$, m is continuous at some such x_S'', and thus $m\left(x_S''\right) \geq m^*\left(x_S''\right)$ by lemma 6, contradiction to (14). So m is continuous on $(0,1)$. Therefore, $\lim_{x_S \to 1^-} m\left(x_S\right) \geq \lim_{x_S \to 1^-} m^*\left(x_S\right) = m^*\left(1\right)$. So m must be continuous at $x_S = 1$. Similar argument shows that m is continuous at $x_S = 0$ if $m\left(0\right) \geq m^*\left(0\right)$, which is the case by the Initial Condition lemma 1. □

Proof of Lemma 4. We first prove the first part. Let $g(x_1, x_2, x_S)$ denote the joint density of $X_{(1)}, X_{(2)}, X_S$. Note that this density satisfies the affiliation property, as does the conditional density $g(x_1, x_2|x_S)$. We then have

$$\begin{aligned}
\frac{F_{(2)}\left(x|x_S\right) - F_{(1)}\left(x|x_S\right)}{f_{(1)}\left(x|x_S\right)} &= \frac{P(X_{(1)} \geq x \geq X_{(2)}|x_S)}{f_{(1)}(x|x_S)} \\
&= \frac{\int_x^1 \int_0^x g(x_1, x_2|x_S)dx_2 dx_1}{\int_0^x g(x, x_2|x_S)dx_2} \\
&= \int_x^1 \frac{\int_0^x g(x_1, x_2, x_S)dx_2}{\int_0^x g(x, x_2, x_S)dx_2} dx_1 \\
&= \int_x^1 \frac{\int_0^x \frac{g(x_1, x_2, x_S)}{g(x, x_2, x_S)} g(x, x_2, x_S)dx_2}{\int_0^x g(x, x_2, x_S)dx_2} dx_1 \\
&= \int_x^1 \mathbb{E}\left(\frac{g(x_1, X_{(2)}, x_S)}{g(x, X_{(2)}, x_S)} \Big| X_{(2)} \leq x\right) dx_1
\end{aligned}$$

For $x_1 \geq x$, affiliation implies

$$\frac{g(x_1, X_{(2)}, x_S)}{g(x, X_{(2)}, x_S)}$$

is increasing in x_S, and therefore the r.h.s. is also increasing in x_S. This proves the first part of the lemma. To prove the second part, we have:

$$\begin{aligned}\frac{f_{(2)}\left(x'|x_S\right)}{f_{(1)}\left(x|x_S\right)} &= \frac{\int_{x'}^{1} g\left(x_1, x', x_S\right) dx_1}{\int_0^x g\left(x, x_2, x_S\right) dx_2} \\ &= \frac{\int_{x'}^{1} \frac{g(x_1,x',x_S)}{g(x,x,x_S)} dx_1}{\int_0^x \frac{g(x,x_2,x_S)}{g(x,x,x_S)} dx_2}\end{aligned}$$

Since $x_1 \geq x' \geq x$, the ratio in the numerator

$$\frac{g\left(x_1, x', x_S\right)}{g\left(x, x, x_S\right)}$$

is increasing in x_S by affiliation, while $x_2 \leq x$ implies that the ratio in the denominator

$$\frac{g\left(x, x_2, x_S\right)}{g\left(x, x, x_S\right)}$$

is decreasing in x_S. We conclude that the r.h.s. is increasing in x_S, which prove the second assertion in the lemma. □

Lemma 5. *$F_{(1)}\left(x|x_S\right), F_{(2)}\left(x|x_S\right), f_{(1)}\left(x|x_S\right), f_{(2)}\left(x|x_S\right)$ are all continuously differentiable on $[0,1]^2$ and $f_{(1)}\left(x|x_S\right)$ is bounded below by a positive number on $(x, x_S) \in [\underline{x}, 1] \times [0,1]$.*

Proof. Let $h\left(x_1, ..., x_n, x_S\right)$ denote the joint probability density function of $X_1, ..., X_n, X_S$. Then the marginal probability density function of X_S is

$$f_{X_S}\left(x_S\right) = \underset{1 \geq x_1 \geq .... \geq x_n \geq 0}{\int \cdots \int} n! h\left(x_1, ..., x_n, x_S\right) dx_1....dx_n.$$

And

$$F_{(1)}\left(x|x_S\right) = \frac{\int_{x_1=0}^{x} \underset{(x_2,...,x_n): x_1 \geq x_2 \geq .... \geq x_n \geq 0}{\int \cdots \int} n! h\left(x_1, x_2, ..., x_n, x_S\right) dx_2...dx_n dx_1}{f_{X_S}\left(x_S\right)}$$

$$F_{(2)}\left(x|x_S\right) = \frac{\int_{x_2=0}^{x} \int_{x_1=x_2}^{1} \underset{(x_3,...,x_n): x_2 \geq x_3 \geq .... \geq x_n \geq 0}{\int \cdots \int} n! h\left(x_1, x_2, x_3, ..., x_n, x_S\right) dx_3...dx_n dx_1 dx_2}{f_{X_S}\left(x_S\right)}$$

so

$$f_{(1)}(x|x_S) = \frac{\displaystyle\underset{(x_2,...,x_n):x\geq x_2\geq....\geq x_n\geq 0}{\int\cdots\int} n!h(x,x_2,...,x_n,x_S)\,dx_2...dx_n}{f_{X_S}(x_S)}$$

$$f_{(2)}(x|x_S) = \frac{\int_{x_1=x}^{1} \displaystyle\underset{(x_3,...,x_n):x\geq x_3\geq....\geq x_n\geq 0}{\int\cdots\int} n!h(x_1,x,x_3,...,x_n,x_S)\,dx_3...dx_n dx_1}{f_{X_S}(x_S)}.$$

By Assumption 1, $h(x_1,...,x_n,x_S)$ has a positive lower bound $\varepsilon > 0$ and positive upper bound $E > \varepsilon$ on $[0,1]^{n+1}$. Since $\underset{1\geq x_1\geq....\geq x_n\geq 0}{\int\cdots\int} n!dx_1...dx_n = \int_{x_1=0}^{1}....\int_{x_n=0}^{1} dx_1...dx_n = 1$, $f_{X_S}(x_S) \in [\varepsilon, E]$ for all $x_S \in [0,1]$. So $F_{(1)}(x|x_S)$, $F_{(2)}(x|x_S)$, $f_{(1)}(x|x_S)$, $f_{(2)}(x|x_S)$ are continuously differentiable on $(x,x_S) \in [0,1]^2$. By the assumption that the optimal full information buyer cutoff $\underline{x} = m^*(0)$ is interior, $f_{(1)}(x|x_S) \geq \frac{\int_{x_2=0}^{x}\cdots\int_{x_n=0}^{x} n\varepsilon dx_2...dx_n}{E} = \frac{n\varepsilon(x)^n}{E} \geq \frac{n\varepsilon(\underline{x})^n}{E} > 0$ for all $(x,x_S) \in [\underline{x},1]\times[0,1]$. □